# The old frequency decomposition problem in the light of new quantization methods *

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UWThPh-1997-21
gr-qc/9708015

**Abstract**

The question is raised whether the unique decomposition of the physical Hilbert space, as emerging in the refined algebraic quantization of a constrained system, may be understood in terms of the old Klein-Gordon type quantization.

*Talk given at the Eighth Marcel Grossmann meeting, Jerusalem, June 22–27, 1997.

# 1 Two quantization methods and the issue of frequency decomposition

The conventional ("old") method to quantize a theory containing a momentum squared constraint is inspired from the situation of a (scalar) particle moving in a space-time background. Restricting ourselves to finite dimensional systems, the wave equation (minisuperspace Wheeler-DeWitt equation) is of the type

$$\mathcal{C}\,\psi \equiv (-\nabla_\alpha \nabla^\alpha + U)\,\psi = 0\,, \tag{1}$$

with $\nabla_\alpha \nabla^\alpha$ being the Laplacian with respect to some Pseudo-Riemannian metric $g_{\alpha\beta}$ on a finite dimensional manifold $\mathcal{M}$, the real function $U$ playing the role of a potential. The space of sufficiently well-behaved solutions admits the well-known indefinite Klein-Gordon type scalar product

$$Q(\psi_1, \psi_2) = -\,\frac{i}{2} \int_\Sigma d\Sigma^\alpha \,(\psi_1^* \overleftrightarrow{\nabla}_\alpha \,\psi_2)\,, \tag{2}$$

where $\Sigma$ is a spacelike hypersurface (with sufficiently regular asymptotic behaviour). If the background structure $(\mathcal{M}, g_{\alpha\beta}, U)$ admits a local symmetry with timelike trajectories (as e.g. for flat $g_{\alpha\beta}$ and constant non-negative $U$, in which case (1) is the Klein-Gordon equation) there is a unique decomposition of wave functions into positive and negative frequency modes ($Q$ restricted to the positive/negative frequency sector being a positive/negative definite scalar product, and the two sectors being orthogonal to each other). In the case of a generic background, it is common folklore that there exists no such unique decomposition [1].

However, there is another quantization scheme for the same sort of systems that starts from the inner product

$$\langle \psi_1, \psi_2 \rangle = \int_{\mathcal{M}} d^n x \,\sqrt{-g}\,\psi_1^* \psi_2 \tag{3}$$

on the Hilbert space of square integrable functions on the manifold. The basic idea of how to proceed dates back to the Sixties (the earliest reference I am aware of is Nachtmann [2]), but there does not seem to have emerged a tradition from that (see however Refs. [3]). After a re-invention of the ansatz in the Nineties [4, 5, 6], this approach has been developed further and has become a viable method by which the quantization of full general relativity is currently being attacked [7]. It runs under several names, the best known being "refined algebraic quantization" (others being "Rieffel induction" and "group averaging").

Without going into the details, I just summarize that (3) gives rise to a positive definite inner product $\langle \,,\, \rangle_{\rm phys}$ on a suitably defined set of solutions of (1). (When inserting two solutions of (1) into (3), one would in general obtain an infinite result, but if the wave operator $\mathcal{C}$ in (1) is self-adjoint, this can be cured by "dividing by

an infinite constant" or, more precise, by averaging over the group generated by $\mathcal{C}$). Thus, upon completion, one ends up with what is usually called the physical Hilbert space $(\mathcal{H}_{\rm phys}, \langle\, ,\, \rangle_{\rm phys})$.

Most researchers employing this new scheme simply forget about the structure (2) that has played an important role in the early years of quantum gravity. Since in quantum gravity or quantum cosmology, (mini)superspace plays a role fundamentally different from the space-time manifold in the particle quantization problem, one may take the point of view that the notion of positive and negative frequency modes does not play any substantial role there. However, the structure (2) still exists, even if it is not payed any attention. (I ignore here the problem that (2) is ill-defined in the full superspace context and must be regularized. In the framework under consideration, $Q$ is well-defined on $\mathcal{H}_{\rm phys}$.)

What can we infer from the fact that $Q$ and $\langle\, ,\, \rangle_{\rm phys}$ *coexist* on one and the same space? The former quantity may be represented in terms of the latter by $Q(\psi_1,\psi_2) = \langle \psi_1, K\psi_2\rangle_{\rm phys}$, where $K$ is a linear (supposedly self-adjoint) operator. In reasonable cases, its (generalized) eigenvalues come in pairs $(-\lambda,\lambda)$ off zero (in the case of the Klein-Gordon equation we have even $K^2=1$), so that the Hilbert space uniquely decomposes as $\mathcal{H}_{\rm phys} = \mathcal{H}^+ \oplus \mathcal{H}^-$. Moreover, $Q$ is positive/negative definite on $\mathcal{H}^\pm$ and the two subspaces are orthogonal to each other with respect to both scalar products. This decomposition has been singled out by the global structure of $(\mathcal{M}, g_{\alpha\beta}, U)$ (note that no local symmetry is necessary for the construction to work). Recently, Hartle and Marolf have exploited the coexistence of the two scalar products, though with different motivation [8].

# 2 Understanding new issues in terms of old methods?

Can the decomposition defined above be viewed as "the correct" identification of positive and negative frequencies? Note that the refined algebraic quantization scheme provides a structure that is in a sense invisible for the Klein-Gordon type approach *although it does not need any additional input*. Perhaps a clarification of this situation could improve our understanding of what happens when we quantize a constrained system. I cannot resolve this puzzle, but I would like to mention a candidate for a procedure defined within the Klein-Gordon framework but transcending the differential geometric setting. It might possibly show us a way how to make contact between the two methods.

In Ref. [9], a framework for treating quite general wave equations of the type (1) with positive potential was proposed. Writing the wave function as $\psi = \chi D e^{iS}$, (with $S$ being a sufficiently globally regular solution of the classical Hamilton-Jacobi equation and $D$ a real function satisfying a certain conservation equation), the wave equation (1) reads $i\,\partial_t \chi = (\frac{1}{2}\,\partial_{tt} + h)\chi$. Here $t=-S$ and $h$ is a linear differential

operator acting tangential to the hypersurfaces $\Sigma_t$ of constant $t$. Although resembling a WKB scheme, no approximation is applied so far. In Ref. [9] it is argued that if an operator $H$ is defined as a series of the type

$$H = h - \frac{1}{2}\,h^2 - \frac{i}{2}\,\dot{h} + \frac{1}{2}\,h^3 + \frac{i}{2}\,\{h, \dot{h}\} - \frac{1}{4}\,\ddot{h} + \ldots \tag{4}$$

(as emerging from the iteration of a certain differential equation), where $\dot{h} \equiv [\partial_t, h]$, then any solution of the Schrödinger equation

$$i\,\partial_t \chi = H\chi \tag{5}$$

solves (1). The actual convergence of (4) seems to depend on the particular background $(\mathcal{M}, g_{\alpha\beta}, U)$, and it is here that some models might be excluded (while remaining intact from the point of view of differential geometry). In case of convergence (which has been checked for the simple cases $h = a + bt + ct^2$, $h = \alpha/t$ and $h = \beta/t^2$ (this last one having applications to FRW quantum cosmology), the set of solutions $\psi$ obtained in this way forms a subspace $\mathcal{H}'^{+}$ which is independent of the choice of the pair $(S, D)$ — called a "WKB-branch" — in which it is calculated and, together with its complex conjugate $\mathcal{H}'^{-}$, decomposes $\mathcal{H}_{\text{phys}}$ into a direct orthogonal sum. $Q$ is positive/negative definite on $\mathcal{H}'^{\pm}$ just as was the case for $\mathcal{H}^{\pm}$ above.

I cannot answer the natural question arising here, whether $\mathcal{H}'^{\pm}$ have something to do (or are even identical) with $\mathcal{H}^{\pm}$ (except for the flat space Klein-Gordon equation, where they *are* identical). Maybe pursuing this route could clarify why the decomposition based on $\mathcal{H}^{\pm}$ is invisible to the Klein-Gordon quantization scheme, at least as long as one remains within the pure differential geometric framework.

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